

# A BASIS FOR THE DIAGONALLY SIGNED-SYMMETRIC POLYNOMIALS

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**ABSTRACT.** Let  $n \geq 1$  be an integer and let  $B_n$  denote the hyperoctahedral group of rank  $n$ . The group  $B_n$  acts on the polynomial ring  $\mathbb{Q}[x_1, \dots, x_n, y_1, \dots, y_n]$  by signed permutations simultaneously on both of the sets of variables  $x_1, \dots, x_n$  and  $y_1, \dots, y_n$ . The invariant ring  $M^{B_n} := \mathbb{Q}[x_1, \dots, x_n, y_1, \dots, y_n]^{B_n}$  is the ring of diagonally signed-symmetric polynomials. In this article we provide an explicit free basis of  $M^{B_n}$  as a module over the ring of symmetric polynomials on both of the sets of variables  $x_1^2, \dots, x_n^2$  and  $y_1^2, \dots, y_n^2$  using signed descent monomials.

## 1. INTRODUCTION

Let  $V$  be an  $n$ -dimensional vector space over a field  $k$  of characteristic zero. Suppose that  $W$  is a finite reflection group in  $V$ ; that is,  $W$  is finite subgroup of  $GL(V)$  generated by elements of order 2 that fix a hyperplane pointwise. Then  $W$  acts by ring automorphisms on the symmetric algebra  $S(V^*)$ , where  $V^*$  is the dual of  $V$ . If we give  $V$  a basis, then  $S(V^*)$  can be identified with a polynomial ring  $k[\mathbf{x}] := k[x_1, \dots, x_n]$ . The action of the group  $W$  on the polynomial ring  $k[\mathbf{x}]$ , under the above identification, has been classically studied. For example, by [7, Theorem A] the ring  $k[\mathbf{x}]^W$  consisting of all  $W$ -invariant polynomials is itself a polynomial ring on  $n$  homogeneous generators. Consider now the diagonal action of  $W$  on the symmetric algebra  $S(V^* \oplus V^*)$ . If we give  $V$  a basis as before, then  $S(V^* \oplus V^*)$  can be identified with a polynomial algebra  $k[\mathbf{x}, \mathbf{y}] := k[x_1, \dots, x_n, y_1, \dots, y_n]$  and  $W$  acts diagonally on it. In this case the ring  $M^W := k[\mathbf{x}, \mathbf{y}]^W$  consisting of all diagonally  $W$ -invariant polynomials is no longer a polynomial algebra. Note that the ring  $R^W := k[\mathbf{x}]^W \otimes k[\mathbf{y}]^W$  of all polynomials that are  $W$ -invariant in both of the sets of variables  $\mathbf{x}$  and  $\mathbf{y}$  is naturally a subring of  $M^W$ . Therefore we can see  $M^W$  as a module over  $R^W$ . It can be seen that in fact  $M^W$  is a free module over  $R^W$  of rank  $|W|$ . This relies on the fact that  $M^W$  is a Cohen-Macaulay ring which is true by [9, Proposition 13]. This article is concerned with the determination of explicit free bases of  $M^W$  as a module over  $R^W$  for a particular class of groups using elementary methods. For simplicity we work with rational coefficients although all the constructions provided here work for any field of characteristic zero.

In [5] Allen provided an explicit basis for the case of the symmetric group. More precisely, suppose that  $W = \Sigma_n$  acts on the polynomial algebra  $\mathbb{Q}[\mathbf{x}] = \mathbb{Q}[x_1, \dots, x_n]$  by permutations of the variables  $x_1, \dots, x_n$ . In this case the invariant ring  $\mathbb{Q}[\mathbf{x}]^{\Sigma_n}$  is the ring of symmetric polynomials which is a polynomial algebra on the elementary symmetric polynomials. Let  $\Sigma_n$  act diagonally on  $\mathbb{Q}[\mathbf{x}, \mathbf{y}] := \mathbb{Q}[x_1, \dots, x_n, y_1, \dots, y_n]$ . Then  $M^{\Sigma_n} = \mathbb{Q}[\mathbf{x}, \mathbf{y}]^{\Sigma_n}$  is the ring of diagonally symmetric or multisymmetric polynomials. Given

$\pi \in \Sigma_n$  define the diagonal descent monomial

$$e_\pi := \prod_{i \in \text{Des}(\pi^{-1})} (x_1 \cdots x_i) \prod_{j \in \text{Des}(\pi)} (y_{\pi(1)} \cdots y_{\pi(j)}) = \prod_{i=1}^n x_i^{d_i(\pi^{-1})} y_i^{d_{\pi^{-1}(i)}(\pi)},$$

where  $\text{Des}(\pi)$  denotes the descent set of  $\pi$  and  $d_i(\pi^{-1})$ ,  $d_{\pi^{-1}(i)}(\pi)$  are integers (see Section 2 for the definitions). Then by [5, Theorem 1.3] the collection  $\{\rho_{\Sigma_n}(e_\pi)\}_{\pi \in \Sigma_n}$  forms a free basis of  $M^{\Sigma_n}$  as a module over  $R^{\Sigma_n} = \mathbb{Q}[\mathbf{x}]^{\Sigma_n} \otimes \mathbb{Q}[\mathbf{y}]^{\Sigma_n}$ , where  $\rho_{\Sigma_n}$  is the averaging operator defined below.

The goal of this article is to show that an analogous construction works for the hyperoctahedral group  $B_n$  acting on the polynomial algebra  $\mathbb{Q}[\mathbf{x}] = \mathbb{Q}[x_1, \dots, x_n]$  by signed permutations. In this case, it is easy to see that invariant ring  $\mathbb{Q}[\mathbf{x}]^{B_n}$  consists of all symmetric polynomials on the variables  $x_1^2, \dots, x_n^2$ . Suppose that  $B_n$  acts diagonally on the polynomial ring  $\mathbb{Q}[\mathbf{x}, \mathbf{y}] = \mathbb{Q}[x_1, \dots, x_n, y_1, \dots, y_n]$  by signed permutations. Then the invariant ring  $M^{B_n} = \mathbb{Q}[\mathbf{x}, \mathbf{y}]^{B_n}$  is the ring of diagonally signed-symmetric polynomials. A free basis of it as a module over  $R^{B_n} = \mathbb{Q}[\mathbf{x}]^{B_n} \otimes \mathbb{Q}[\mathbf{y}]^{B_n}$  can be constructed in the same spirit as in the case of permutations. Given  $\sigma \in B_n$ , define the diagonal signed descent monomial

$$c_\sigma := \left( \prod_{i=1}^n x_i^{f_i(\sigma^{-1})} \right) \left( \prod_{i=1}^n y_{|\sigma(i)|}^{f_i(\sigma)} \right).$$

See Section 3 for the definition of the numbers  $f_i(\sigma)$ . The goal of this article is the following theorem.

**Theorem 1.1.** *Suppose that  $n \geq 1$ . Then the collection  $\{\rho(c_\sigma)\}_{\sigma \in B_n}$  forms a free basis of  $M^{B_n}$  as a module over  $R^{B_n}$ , where  $\rho$  is the averaging operator.*

A similar basis to the one given in the previous theorem was constructed in [6]. Moreover, in there a nice combinatorial interpretation of the basis monomials was provided in terms of certain diagrams of the square lattice. The author would like to thank F. Bergeron and R. Biagioli for pointing out their work to him.

## 2. THE SYMMETRIC GROUP

In this section we provide a brief review of an explicit basis for the coinvariant ring for groups of type  $A$  using descent monomials constructed by Garsia and Stanton in [8]. A construction of a free basis for the ring of diagonally symmetric polynomials as a module over the symmetric polynomials constructed by Allen in [5] is also reviewed.

**2.1. Major index.** For every integer  $n \geq 1$ , let  $\Sigma_n$  denote the symmetric group of self bijections of the set  $\{1, 2, \dots, n\}$ . We use the notation  $\pi = [\pi_1, \dots, \pi_n]$  for an element  $\pi \in \Sigma_n$  with  $\pi_i = \pi(i)$  for  $1 \leq i \leq n$ . Given  $\pi \in \Sigma_n$  define its descent to be the set

$$\text{Des}(\pi) := \{1 \leq i \leq n-1 \mid \pi(i) > \pi(i+1)\}$$

and for  $1 \leq i \leq n$  let

$$d_i(\pi) := |\{j \in \text{Des}(\pi) \mid j \geq i\}|.$$

The numbers  $d_i(\pi)$  clearly satisfy the following properties:

- (1)  $d_1(\pi) \geq d_2(\pi) \geq \cdots \geq d_{n-1}(\pi) \geq d_n(\pi) = 0$ , and

(2) if  $i < j$  and  $d_i(\pi) = d_j(\pi)$  then  $\pi(i) < \pi(i+1) < \cdots < \pi(j)$ .

The major index of  $\pi \in \Sigma_n$ , denoted by  $\text{maj}(\pi)$ , is defined to be

$$\text{maj}(\pi) := \sum_{i=1}^n d_i(\pi) = \sum_{i \in \text{Des}(\pi)} i.$$

In [10], MacMahon showed that this statistic is equidistributed with respect to the length function; that is, the number of permutations of length  $n$  with  $k$  inversions is the same as the number of permutations of length  $n$  with major index equal to  $k$ . Note that the numbers  $d_1(\pi) \geq d_2(\pi) \geq \cdots \geq d_{n-1}(\pi)$  are defined exactly to provide a partition of the integer  $\text{maj}(\pi)$ .

**2.2. Descent monomials.** Suppose that  $\mathbf{x} = \{x_1, \dots, x_n\}$  is a set of algebraically independent commuting variables. Consider the polynomial algebra  $\mathbb{Q}[\mathbf{x}] := \mathbb{Q}[x_1, \dots, x_n]$  seen as a graded ring with  $\deg(x_i) = 1$  for  $1 \leq i \leq n$ . The group  $\Sigma_n$  acts naturally on the polynomial algebra  $\mathbb{Q}[\mathbf{x}]$  by permuting the variables  $x_1, \dots, x_n$ . It is well known that the ring of  $\Sigma_n$ -invariants,  $\mathbb{Q}[\mathbf{x}]^{\Sigma_n}$ , is a polynomial algebra on the generators  $e_1(x_1, \dots, x_n), \dots, e_n(x_1, \dots, x_n)$ , where  $e_k(x_1, \dots, x_n)$  is the  $k$ -th elementary symmetric polynomial

$$e_k(x_1, \dots, x_n) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} x_{i_1} x_{i_2} \cdots x_{i_k}.$$

Suppose that  $\pi \in \Sigma_n$ . Define the descent monomial associated to  $\pi$  to be

$$a_\pi := \prod_{i \in \text{Des}(\pi)} x_{\pi(i)} \cdots x_{\pi(i)} = \prod_{i=1}^n x_{\pi(i)}^{d_i(\pi)} = \prod_{i=1}^n x_i^{d_{\pi^{-1}(i)}(\pi)}.$$

**Example:** Suppose that  $\pi = [6, 2, 1, 4, 3, 5]$ . Then  $a_\pi = x_1 x_2^2 x_4 x_6^3$ .

Note that  $\deg(a_\pi) = \text{maj}(\pi)$ . In [8] Garsia and Stanton used Stanley–Reisner rings to show that these monomials provide a basis for the coinvariant algebra of type  $A$ . More precisely, let  $I_n^A$  denote the ideal in  $\mathbb{Q}[\mathbf{x}]$  generated by the symmetric polynomials  $e_1(x_1, \dots, x_n), \dots, e_n(x_1, \dots, x_n)$ . Then  $\mathbb{Q}[\mathbf{x}]/I_n^A$  is the coinvariant algebra of type  $A$ . Let  $\bar{a}_\pi$  denote the image of  $a_\pi$  in the coinvariant algebra under the natural map. In [8] it was proved that the collection  $\{\bar{a}_\pi\}_{\pi \in \Sigma_n}$  forms a basis of  $\mathbb{Q}[\mathbf{x}]/I_n^A$  as a  $\mathbb{Q}$ -vector space. Moreover, the collection  $\{a_\pi\}_{\pi \in \Sigma_n}$  provides a free basis for  $\mathbb{Q}[\mathbf{x}]$  as a module over the symmetric polynomials  $\mathbb{Q}[\mathbf{x}]^{\Sigma_n}$ . This result has an interesting geometric application. Consider the flag manifold  $U(n)/T$ , where  $T \subset U(n)$  is a maximal torus. Then  $H^*(U(n)/T; \mathbb{Q})$  can be identified with the invariant algebra  $\mathbb{Q}[\mathbf{x}]/I_n^A$ , but under this identification we need to graduate the variables  $x_1, \dots, x_n$  with degree 2. Therefore the descent monomials provide an explicit basis for the cohomology of the flag manifold  $U(n)/T$ .

**2.3. Diagonal descent monomials.** Let  $\mathbf{y} = \{y_1, \dots, y_n\}$  be another set of algebraically independent commuting variables of degree 1 and consider the polynomial algebra  $\mathbb{Q}[\mathbf{x}, \mathbf{y}]$ . The symmetric group  $\Sigma_n$  acts diagonally on this polynomial ring and the ring of  $\Sigma_n$ -invariants,  $M^{\Sigma_n} := \mathbb{Q}[\mathbf{x}, \mathbf{y}]^{\Sigma_n}$ , is known as the ring of diagonally symmetric or multisymmetric polynomials. Note that the ring of polynomials that are symmetric in both the variables  $\mathbf{x}$  and  $\mathbf{y}$ ,  $R^{\Sigma_n} := \mathbb{Q}[\mathbf{x}]^{\Sigma_n} \otimes \mathbb{Q}[\mathbf{y}]^{\Sigma_n}$ , is a subring of  $M^{\Sigma_n}$  and thus  $M^{\Sigma_n}$  can be

seen as a module over  $R^{\Sigma_n}$ . In [5] Allen constructed a free basis for the module  $M^{\Sigma_n}$  using a variation of the descent monomials. Given  $\pi \in \Sigma_n$  define the diagonal descent monomial to be

$$e_\pi := \left( \prod_{i=1}^n x_i^{d_i(\pi^{-1})} \right) \left( \prod_{i=1}^n y_{\pi(i)}^{d_i(\pi)} \right) = \prod_{i=1}^n x_i^{d_i(\pi^{-1})} y_i^{d_{\pi^{-1}(i)}(\pi)}.$$

**Example:** Suppose that  $\pi = [4, 6, 1, 2, 5, 3]$ . Then  $e_\pi = x_1^2 x_2^2 x_3^2 x_4 x_5 y_1 y_2 y_4^2 y_5 y_6^2$ .

Note that the total degree of  $e_\pi$  is given by  $\deg(e_\pi) = \text{maj}(\pi) + \text{maj}(\pi^{-1})$  for every  $\pi \in \Sigma_n$ . Consider the averaging operator

$$\begin{aligned} \rho_{\Sigma_n} : \mathbb{Q}[\mathbf{x}, \mathbf{y}] &\rightarrow \mathbb{Q}[\mathbf{x}, \mathbf{y}]^{\Sigma_n} = M^{\Sigma_n} \\ f &\mapsto \frac{1}{n!} \sum_{\pi \in \Sigma_n} \pi \cdot f. \end{aligned}$$

Thus by definition  $\rho_{\Sigma_n}(e_\pi)$  is a diagonally symmetric polynomial. By [5, Theorem 1.3] the collection  $\{\rho_{\Sigma_n}(e_\pi)\}_{\pi \in \Sigma_n}$  forms a free basis of  $M^{\Sigma_n}$  as a module over  $R^{\Sigma_n}$ . It turns out that this result also has an interesting geometric application. Let  $B(2, U(n))$  be the geometric realization of the simplicial space obtained by defining  $B_k(2, U(n)) = \text{Hom}(\mathbb{Z}^k, U(n))$ , where  $\text{Hom}(\mathbb{Z}^k, U(n))$  is the space of ordered commuting  $k$ -tuples in  $U(n)$ . In [1] it is proved that the diagonal descent monomials can be used to obtain an explicit basis of  $H^*(B(2, U(n)); \mathbb{Q})$  seen as a module over  $H^*(BU(n); \mathbb{Q})$ , where  $BU(n)$  is the classifying space of  $U(n)$ .

### 3. THE HYPEROCTAHEDRAL GROUP

In this section we provide analogue constructions to the ones presented in the previous section, where the symmetric group is replaced by the group of signed permutations.

**3.1. Flag major index.** Suppose that  $n \geq 1$  is an integer. Denote by  $\mathbb{I}_n$  the set of integers between  $-n$  and  $n$  not including 0; that is,

$$\mathbb{I}_n := \{-n, -n+1, \dots, -1, 1, \dots, n-1, n\}.$$

Let  $B_n$  denote the group of bijections  $\sigma : \mathbb{I}_n \rightarrow \mathbb{I}_n$  such that  $\sigma(-k) = -\sigma(k)$  for all  $k \in \mathbb{I}_n$ , with the composition of functions as the group operation. Thus the group  $B_n$  is the group of signed permutations, also known as the hyperoctahedral group of rank  $n$ . It is easy to see that  $B_n$  is isomorphic to the semidirect product  $\Sigma_n \ltimes (\mathbb{Z}/2)^n$ . We use the following notation for elements  $\sigma \in B_n$ . Let  $\sigma_i = \sigma(i)$  for  $1 \leq i \leq n$ , then we write  $\sigma = [\sigma_1, \dots, \sigma_n]$ . Note that the group  $B_n$  is the Weyl group associated to Lie groups of type  $B_n$  and  $C_n$  and that the symmetric group  $\Sigma_n$  is naturally a subgroup of  $B_n$ . As in the case of the symmetric group, given  $\sigma \in B_n$  define its descent to be the set

$$\text{Des}(\sigma) := \{1 \leq i \leq n-1 \mid \sigma(i) > \sigma(i+1)\}$$

and for  $1 \leq i \leq n$  let

$$d_i(\sigma) := |\{j \in \text{Des}(\sigma) \mid j \geq i\}|.$$

As before the numbers  $d_i(\sigma)$  satisfy the following important properties:

- (1)  $d_1(\sigma) \geq d_2(\sigma) \geq \dots \geq d_{n-1}(\sigma) \geq d_n(\sigma) = 0$ , and

(2) if  $i < j$  and  $d_i(\sigma) = d_j(\sigma)$  then  $\sigma(i) < \sigma(i+1) < \dots < \sigma(j)$ .

On the other hand, define

$$\varepsilon_i(\sigma) := \begin{cases} 0 & \text{if } \sigma(i) > 0, \\ 1 & \text{if } \sigma(i) < 0, \end{cases}$$

and

$$f_i(\sigma) := 2d_i(\sigma) + \varepsilon_i(\sigma).$$

It is easy to see that the numbers  $f_i(\sigma)$  also satisfy the properties:

- (1)  $f_1(\sigma) \geq f_2(\sigma) \geq \dots \geq f_n(\sigma)$ , and
- (2) if  $i < j$  and  $f_i(\sigma) = f_j(\sigma)$  implies  $\sigma(i) < \sigma(i+1) < \dots < \sigma(j)$  and all of these numbers have the same sign.

The flag major index of  $\sigma \in B_n$ , denoted by  $\text{fmaj}(\sigma)$  is defined to be

$$\text{fmaj}(\sigma) := \sum_{i=1}^n f_i(\sigma) = 2 \text{maj}(\sigma) + \text{neg}(\sigma),$$

where  $\text{maj}(\sigma) = \sum_{i \in \text{Des}(\sigma)} i$  is the major index of  $\sigma$  and  $\text{neg}(\sigma) = |\{1 \leq i \leq n \mid \sigma(i) < 0\}|$ . This statistic for elements in  $B_n$  was introduced in [4] and further studied in [2], [3] as a generalization of the major index for the hyperoctahedral group. This tool has successfully been used to study representation theoretical properties of the group  $B_n$  (see for example [3]). Note that the numbers  $f_1(\sigma) \geq \dots \geq f_n(\sigma)$  provide a partition of the flag major index  $\text{fmaj}(\sigma)$  in a similar way as in the case of the major index of a permutation in  $\Sigma_n$ . Moreover, if  $\sigma \in \Sigma_n$  then  $\text{fmaj}(\sigma) = 2 \text{maj}(\sigma)$  so the flag major index is indeed a natural generalization of the major index.

**3.2. Signed descent monomials.** Suppose that  $\mathbf{x} = \{x_1, \dots, x_n\}$  is a set of algebraically independent commuting variables. Consider the polynomial algebra  $\mathbb{Q}[\mathbf{x}]$  seen as a graded ring with  $\deg(x_i) = 1$  for  $1 \leq i \leq n$ . The group  $B_n$  acts naturally on the polynomial algebra  $\mathbb{Q}[\mathbf{x}]$  by degree preserving ring homomorphisms in the following way. If  $\sigma \in B_n$  then

$$\sigma \cdot (x_1^{i_1} \cdots x_n^{i_n}) := \left( \frac{\sigma(1)}{|\sigma(1)|} \right)^{i_1} \cdots \left( \frac{\sigma(n)}{|\sigma(n)|} \right)^{i_n} x_{|\sigma(1)|}^{i_1} \cdots x_{|\sigma(n)|}^{i_n}$$

Thus each  $\sigma$  permutes the variables  $x_1, \dots, x_n$  with a suitable sign change. It is easy to see that any polynomial in the ring of  $B_n$ -invariants,  $\mathbb{Q}[\mathbf{x}]^{B_n}$ , must be a symmetric polynomial in the variables  $x_1^2, \dots, x_n^2$ . It follows that  $\mathbb{Q}[\mathbf{x}]^{B_n}$  is a polynomial algebra on the symmetric polynomials  $e_1(x_1^2, \dots, x_n^2), \dots, e_n(x_1^2, \dots, x_n^2)$ . Suppose that  $\sigma \in B_n$ . Define the signed descent monomial

$$b_\sigma := \prod_{i=1}^n x_{|\sigma(i)|}^{f_i(\sigma)} = \prod_{i=1}^n x_i^{f_{|\sigma^{-1}(i)|}(\sigma)}.$$

**Example:** Suppose that  $\sigma = [-6, 2, -1, -4, 3, 5]$ . Then  $b_\sigma = x_1^3 x_2^4 x_4 x_5^5$ .

Note that  $\deg(b_\sigma) = \text{fmaj}(\sigma)$  for every  $\sigma \in B_n$ . The signed descent monomials can be used to obtain a basis for the coinvariant algebra for groups of type  $B, C$ . More precisely, let  $I_n^B$  denote the ideal in  $\mathbb{Q}[\mathbf{x}]$  generated by the elements  $e_1(x_1^2, \dots, x_n^2), \dots, e_n(x_1^2, \dots, x_n^2)$ .

Then the quotient  $\mathbb{Q}[\mathbf{x}]/I_n^B(\mathbf{x})$  is the coinvariant algebra in this case. Let  $\bar{b}_\sigma$  denote the image of  $b_\sigma$  in the coinvariant algebra under the natural map. By [3, Corollary 5.3] the collection  $\{\bar{b}_\sigma\}_{\sigma \in B_n}$  forms a basis of  $\mathbb{Q}[\mathbf{x}]/I_n^B(\mathbf{x})$  as a  $\mathbb{Q}$ -vector space. We can also see  $\mathbb{Q}[\mathbf{x}]$  as a module over the invariant ring  $\mathbb{Q}[\mathbf{x}]^{B_n}$ . Since  $\{\bar{b}_\sigma\}_{\sigma \in B_n}$  forms a basis of  $\mathbb{Q}[\mathbf{x}]/I_n^B(\mathbf{x})$  as a  $\mathbb{Q}$ , then using [5, Theorem 1.2] it can be seen that  $\{b_\sigma\}_{\sigma \in B_n}$  forms a free basis of  $\mathbb{Q}[\mathbf{x}]$  as a module over  $\mathbb{Q}[\mathbf{x}]^{B_n}$ . This result has a geometric application as in the case of the symmetric group, namely, the signed descent monomials provide an explicit basis for the rational cohomology of the flag manifold  $G/T$ , for a compact connected Lie group  $G$  of type  $B_n, C_n$  and a maximal torus  $T \subset G$ .

**3.3. Diagonal signed descent monomials.** Consider now  $\mathbf{y} = \{y_1, \dots, y_n\}$  another set of algebraically independent commuting variables of degree 1 and consider the polynomial algebra  $\mathbb{Q}[\mathbf{x}, \mathbf{y}] := \mathbb{Q}[\mathbf{x}] \otimes \mathbb{Q}[\mathbf{y}] = \mathbb{Q}[x_1, \dots, x_n, y_1, \dots, y_n]$ . The group  $B_n$  acts diagonally on this polynomial ring; that is,  $B_n$  acts as signed permutations simultaneously on the variables  $x_1, \dots, x_n$  and  $y_1, \dots, y_n$ . Define  $M^{B_n} := \mathbb{Q}[\mathbf{x}, \mathbf{y}]^{B_n}$ . In other words  $M^{B_n}$  is the ring of diagonally signed-symmetric polynomials. Note that the ring of polynomials that are signed-symmetric on both the variables  $\mathbf{x}$  and  $\mathbf{y}$ ,  $R^{B_n} := \mathbb{Q}[\mathbf{x}]^{B_n} \otimes \mathbb{Q}[\mathbf{y}]^{B_n}$ , is a subring of  $M^{B_n}$  and thus we can see  $M^{B_n}$  as a module over  $R^{B_n}$ . As it was pointed out before  $M^{B_n}$  is a free module over  $R^{B_n}$  and the goal of this article is to construct an explicit basis for  $M^{B_n}$  as a module over  $R^{B_n}$ . For this we will consider the following monomials.

**Definition 3.1.** Suppose that  $\sigma \in B_n$ . The diagonal signed descent monomial associated to  $\sigma$  is defined to be

$$c_\sigma := \left( \prod_{i=1}^n x_i^{f_i(\sigma^{-1})} \right) \left( \prod_{i=1}^n y_{|\sigma(i)|}^{f_i(\sigma)} \right) = \prod_{i=1}^n x_i^{f_i(\sigma^{-1})} y_i^{f_{|\sigma^{-1}(i)|}(\sigma)}.$$

Note that for any  $\sigma \in B_n$  we have  $\deg(\sigma) = \text{fmaj}(\sigma) + \text{fmaj}(\sigma^{-1})$ .

**Example:** Suppose that  $\sigma = [2, -1, -4, 3]$ . Then  $c_\sigma = x_1^3 x_2^2 x_3^2 x_4 y_1^3 y_2^4 y_4$ .

**3.4. Averaging polynomials.** Consider the averaging operator

$$\rho : \mathbb{Q}[\mathbf{x}, \mathbf{y}] \rightarrow \mathbb{Q}[\mathbf{x}, \mathbf{y}]^{B_n} = M^{B_n}$$

$$f \mapsto \frac{1}{|B_n|} \sum_{\sigma \in B_n} \sigma \cdot f.$$

The map  $\rho$  is a ring homomorphism that is surjective. Moreover, as a  $\mathbb{Q}$ -vector space  $M^{B_n}$  is generated by elements of the form  $\rho(m(\mathbf{x}, \mathbf{y}))$ , where  $m(\mathbf{x}, \mathbf{y})$  is a monomial in  $\mathbb{Q}[\mathbf{x}, \mathbf{y}]$ . We will use the following notation. Suppose that  $p = (p_1, \dots, p_n)$  is a sequence of non-negative integers. Then we write  $x^p$  to denote the monomial  $x_1^{p_1} \cdots x_n^{p_n}$ .

**Lemma 3.2.** Suppose that  $p = (p_1, \dots, p_n)$  and  $q = (q_1, \dots, q_n)$  are sequences of non-negative integers and let  $m(\mathbf{x}, \mathbf{y}) = x^p y^q$ . If  $p_k + q_k$  is odd for some  $1 \leq k \leq n$  then  $\rho(m(\mathbf{x}, \mathbf{y})) = 0$ .

**Proof:** Suppose that  $1 \leq k \leq n$  is such that  $p_k + q_k$  is odd. Define

$$B_n^+ = \{\sigma \in B_n \mid \sigma(k) > 0\} \text{ and } B_n^- = \{\sigma \in B_n \mid \sigma(k) < 0\}.$$

Note that  $B_n = B_n^+ \sqcup B_n^-$ . Moreover, there is a bijection  $\tau : B_n^+ \rightarrow B_n^-$  defined by

$$\tau(\sigma)(i) := \begin{cases} \sigma(i) & \text{if } i \neq k, \\ -\sigma(i) & \text{if } i = k. \end{cases}$$

By definition

$$\rho(m(\mathbf{x}, \mathbf{y})) = \frac{1}{|B_n|} \sum_{\sigma \in B_n} c_\sigma x_{|\sigma(1)|}^{p_1} \cdots x_{|\sigma(n)|}^{p_n} y_{|\sigma(1)|}^{q_1} \cdots y_{|\sigma(n)|}^{q_n},$$

where

$$c_\sigma = \left( \frac{\sigma(1)}{|\sigma(1)|} \right)^{p_1+q_1} \cdots \left( \frac{\sigma(n)}{|\sigma(n)|} \right)^{p_n+q_n}.$$

For any  $\sigma \in B_n^+$  we have  $c_{\tau(\sigma)} = -c_\sigma$  since  $i_k + j_k$  is odd. Therefore

$$\begin{aligned} \rho(m(\mathbf{x}, \mathbf{y})) &= \frac{1}{|B_n|} \sum_{\sigma \in B_n} c_\sigma x_{|\sigma(1)|}^{p_1} \cdots x_{|\sigma(n)|}^{p_n} y_{|\sigma(1)|}^{q_1} \cdots y_{|\sigma(n)|}^{q_n} \\ &= \frac{1}{|B_n|} \sum_{\sigma \in B_n^+} (c_\sigma + c_{\tau(\sigma)}) x_{|\sigma(1)|}^{p_1} \cdots x_{|\sigma(n)|}^{p_n} y_{|\sigma(1)|}^{q_1} \cdots y_{|\sigma(n)|}^{q_n} = 0. \end{aligned}$$

□

By the previous lemma, it follows that  $M^{B_n}$  is generated as a vector space over  $\mathbb{Q}$  by the elements of the form  $\rho(m(\mathbf{x}, \mathbf{y}))$ , where  $m(\mathbf{x}, \mathbf{y}) = x^p y^q$  and  $p = (p_1, \dots, p_n)$  and  $q = (q_1, \dots, q_n)$  are sequences of integers such that  $p_k + q_k$  is even for all  $1 \leq k \leq n$ . Note that for any such monomial we have

$$\rho(m(\mathbf{x}, y)) = \frac{1}{n!} \sum_{\alpha \in \Sigma_n} x_{\alpha(1)}^{p_1} \cdots x_{\alpha(n)}^{p_n} y_{\alpha(1)}^{q_1} \cdots y_{\alpha(n)}^{q_n}$$

and thus  $\rho(m(\mathbf{x}, \mathbf{y})) \neq 0$  for such monomials. Suppose that  $\sigma \in B_n$  and consider the diagonal signed descent monomial  $c_\sigma = \prod_{i=1}^n x_i^{f_i(\sigma^{-1})} y_i^{f_{|\sigma^{-1}(i)|}(\sigma)}$  as defined above. Using the properties of the numbers  $f_i(\sigma)$ , it is easy to see that for every  $1 \leq i \leq n$  the numbers  $f_i(\sigma^{-1})$  and  $f_{|\sigma^{-1}(i)|}(\sigma)$  have the same parity. Therefore by the previous comment we have  $\rho(c_\sigma) \neq 0$  for all  $\sigma \in B_n$ . By definition  $\rho(c_\sigma) \in M^{B_n}$ . We will show below that the collection  $\{\rho(c_\sigma)\}_{\sigma \in B_n}$  forms a free basis of  $M^{B_n}$  as a module over  $R^{B_n}$ .

**3.5. Ordering of monomials.** We will work with monomials  $m(\mathbf{x}, \mathbf{y})$  whose exponents are ordered in the following way.

**Definition 3.3.** Suppose that  $p = (p_1, \dots, p_n)$  and  $q = (q_1, \dots, q_n)$  are two sequences of non-negative integers with  $p_k + q_k$  even for all  $1 \leq k \leq n$ . We say that the monomial  $m(\mathbf{x}, \mathbf{y}) = x^p y^q$  is ordered and write  $m(\mathbf{x}, \mathbf{y}) \in \mathcal{O}_n$  if the exponents of  $m(\mathbf{x}, \mathbf{y})$  satisfy the following conditions:

- (1)  $p_1 \geq p_2 \geq \cdots \geq p_n$ ,
- (2) if  $p_i = p_{i+1}$  and they are even, then  $q_i \geq q_{i+1}$ , and
- (3) if  $p_i = p_{i+1}$  and they are odd, then  $q_i \leq q_{i+1}$ .

The previous ordering can be described in the following way. For each integer  $q$  define

$$\mathfrak{s}(q) := \begin{cases} q & \text{if } q \text{ is even,} \\ -q & \text{if } q \text{ is odd.} \end{cases}$$

Then  $m(\mathbf{x}, \mathbf{y}) = x^p y^q \in \mathcal{O}_n$  if and only if  $(p_1, \mathfrak{s}(q_1)) \geq_\ell \cdots \geq_\ell (p_n, \mathfrak{s}(q_n))$ , where  $\geq_\ell$  denotes the lexicographic order. Using the definition given above it is easy to see that for every  $\sigma \in B_n$  the diagonal signed descent monomial  $c_\sigma$  is ordered in this way; that is,  $c_\sigma \in \mathcal{O}_n$  for all  $\sigma \in B_n$ .

Suppose now that  $m(\mathbf{x}, \mathbf{y}) = x^p y^q$  is a monomial with  $p_k + q_k$  even for all  $1 \leq k \leq n$  but whose exponents are not necessarily ordered as above. Consider the signed-symmetric polynomial  $\rho(m(\mathbf{x}, \mathbf{y}))$ . Note that in this polynomial there exists a unique monomial  $n(\mathbf{x}, \mathbf{y})$  whose exponents are ordered as above; that is,  $\rho(m(\mathbf{x}, \mathbf{y}))$  contains a unique monomial  $n(\mathbf{x}, \mathbf{y}) \in \mathcal{O}_n$  and  $\rho(m(\mathbf{x}, \mathbf{y})) = \rho(n(\mathbf{x}, \mathbf{y}))$ . Because of this we can restrict ourselves to monomials  $m(\mathbf{x}, \mathbf{y}) \in \mathcal{O}_n$ .

Next we define a total order on  $\mathcal{O}_n$ . For this suppose that  $q = (q_1, \dots, q_n)$  is a sequence of integers. Define the ordering of  $q$  to be

$$\mathfrak{o}(q) := (q_{\alpha(1)}, \dots, q_{\alpha(n)}),$$

where  $(q_{\alpha(1)}, \dots, q_{\alpha(n)})$  is a rearrangement of the sequence  $q$  in a decreasing way; that is,  $\alpha \in \Sigma_n$  is such that  $q_{\alpha(1)} \geq \cdots \geq q_{\alpha(n)}$ . Suppose now that  $m(\mathbf{x}, \mathbf{y}) = x^p y^q$  is a monomial with  $p = (p_1, \dots, p_n)$  and  $q = (q_1, \dots, q_n)$ . Define

$$\mathfrak{o}(m(\mathbf{x}, \mathbf{y})) := (\mathfrak{o}(p), \mathfrak{o}(q)).$$

In other words,  $\mathfrak{o}(m(\mathbf{x}, \mathbf{y}))$  recovers the exponents of the monomial  $m(\mathbf{x}, \mathbf{y})$  ordered in a decreasing fashion. Using this we can define the following total order on  $\mathcal{O}_n$ .

**Definition 3.4.** Suppose that  $m(\mathbf{x}, \mathbf{y}) = x^p y^q$  and  $n(\mathbf{x}, \mathbf{y}) = x^i y^j$  are two monomials in  $\mathcal{O}_n$ . We write  $m(\mathbf{x}, \mathbf{y}) \succcurlyeq n(\mathbf{x}, \mathbf{y})$  if and only if

- (1)  $\mathfrak{o}(m(\mathbf{x}, \mathbf{y})) \geq_\ell \mathfrak{o}(n(\mathbf{x}, \mathbf{y}))$ , and
- (2) if  $\mathfrak{o}(m(\mathbf{x}, \mathbf{y})) = \mathfrak{o}(n(\mathbf{x}, \mathbf{y}))$  then  $(p, \mathfrak{s}(q)) \geq_\ell (i, \mathfrak{s}(j))$ .

Here for a sequence of integers  $q = (q_1, \dots, q_n)$  we have  $\mathfrak{s}(q) := (\mathfrak{s}(q_1), \dots, \mathfrak{s}(q_n))$ .

**Example:** Suppose that  $n = 4$ . Consider the monomials  $m(\mathbf{x}, \mathbf{y}) = x_1^7 x_2^6 x_3^6 x_4^5 y_1^3 y_2^8 y_3^6 y_4^5$  and  $n(\mathbf{x}, \mathbf{y}) = x_1^7 x_2^6 x_3^5 y_1^5 y_2^8 y_3^6 y_4^3$ . Then  $m(\mathbf{x}, \mathbf{y}), n(\mathbf{x}, \mathbf{y}) \in \mathcal{O}_4$  are such that

$$\mathfrak{o}(m(\mathbf{x}, \mathbf{y})) = \mathfrak{o}(n(\mathbf{x}, \mathbf{y})) = (7, 6, 6, 5, 8, 6, 5, 3)$$

and  $m(\mathbf{x}, \mathbf{y}) \succcurlyeq n(\mathbf{x}, \mathbf{y})$ .

**3.6. Signed index permutation.** Suppose that  $m(\mathbf{x}, \mathbf{y}) = x^p y^q \in \mathcal{O}_n$ . Note that by construction the sequence  $p = (p_1, \dots, p_n)$  is ordered in a decreasing way. This is not necessarily true for the sequence  $q = (q_1, \dots, q_n)$ . With this in mind, we can associate to the monomial  $m(\mathbf{x}, \mathbf{y})$  the unique element  $\sigma \in B_n$ , which we call its signed index permutation, that satisfies the following properties:

- (1)  $q_{|\sigma(1)|} \geq q_{|\sigma(2)|} \geq \cdots \geq q_{|\sigma(n)|}$ ,
- (2) if  $0 < i < j$  and  $q_{|\sigma(i)|} = q_{|\sigma(j)|}$  then  $\sigma(i) < \sigma(i+1) < \cdots < \sigma(j)$ , and
- (3)  $q_{|\sigma(i)|}$  is even if and only if  $\sigma(i) > 0$ .



In other words, the signed permutation  $\sigma$  is the unique element in  $B_n$  whose signs are determined by the parity of the  $q_i$ 's and that orders the elements in the sequence  $q = (q_1, \dots, q_n)$  in decreasing way breaking ties from left to right for even values of  $q_i$  and from right to left for odd values of  $q_i$ .

**Example:** Suppose  $m(\mathbf{x}, \mathbf{y}) = x_1^7 x_2^6 x_3^6 x_4^5 x_5^5 x_6^3 y_1^3 y_2^8 y_3^6 y_4^3 y_5^5 y_6^5$ . Then the signed index permutation associated to  $m(\mathbf{x}, \mathbf{y})$  is  $\sigma = [2, 3, -6, -5, -4, -1]$ .

**3.7. Exponent decomposition.** Suppose that  $m(\mathbf{x}, \mathbf{y}) = x^p y^q$  is a monomial in  $\mathcal{O}_n$  and let  $\sigma \in B_n$  be the signed index permutation associated to  $m(\mathbf{x}, \mathbf{y})$  as explained above. We can use the signed permutation  $\sigma$  to obtain a decomposition of the sequences  $p$  and  $q$  as we explain next. We start by decomposing  $q$ .

**Property 3.5.** The sequence  $\{q_{|\sigma(i)|} - f_i(\sigma)\}_{i=1}^n$  is a decreasing sequence of non-negative even integers.

**Proof:** By definition and property (3) of the signed index permutation  $\sigma$  we have

$$\begin{aligned} q_{|\sigma(i)|} - f_i(\sigma) &= q_{|\sigma(i)|} - 2d_i(\sigma) - \varepsilon_i(\sigma) \\ &\equiv q_{|\sigma(i)|} - \varepsilon_i(\sigma) \pmod{2} \\ &\equiv 0 \pmod{2}. \end{aligned}$$

This proves that  $q_{|\sigma(i)|} - f_i(\sigma)$  is even for all  $1 \leq i \leq n$ . Note that  $q_{|\sigma(n)|} - f_n(\sigma) = q_{|\sigma(n)|} - \varepsilon_n(\sigma) \geq 0$  since  $q_{|\sigma(n)|} \geq 0$  and  $q_{|\sigma(n)|}$  and  $\varepsilon_n(\sigma)$  have the same parity. It remains to prove  $q_{|\sigma(i)|} - f_i(\sigma) \geq q_{|\sigma(i+1)|} - f_{i+1}(\sigma)$  for all  $1 \leq i \leq n-1$ . For this we consider the following cases.

- Case 1. Suppose that  $\sigma(i) < \sigma(i+1)$ . This implies  $d_i(\sigma) = d_{i+1}(\sigma)$ . Thus in this case we need to prove that  $q_{|\sigma(i)|} - \varepsilon_i(\sigma) \geq q_{|\sigma(i+1)|} - \varepsilon_{i+1}(\sigma)$ . Since  $q_{|\sigma(i)|} \geq q_{|\sigma(i+1)|}$  the only case we need to inspect is the case  $\varepsilon_i(\sigma) = 1$  and  $\varepsilon_{i+1}(\sigma) = 0$ . However, under this assumption  $q_{|\sigma(i)|}$  is odd and  $q_{|\sigma(i+1)|}$  is even and thus  $q_{|\sigma(i)|} - 1 \geq q_{|\sigma(i+1)|}$ .

- Case 2. Suppose that  $\sigma(i) > \sigma(i+1)$  and  $\varepsilon_i(\sigma) \neq \varepsilon_{i+1}(\sigma)$ . This implies  $\varepsilon_i(\sigma) = 0$  and  $\varepsilon_{i+1}(\sigma) = 1$ . We have  $d_i(\sigma) = d_{i+1}(\sigma) + 1$ . In this case we need to show that  $q_{|\sigma(i)|} - 1 \geq q_{|\sigma(i+1)|}$ . Note that  $q_{|\sigma(i)|}$  must be even and  $q_{|\sigma(i+1)|}$  must be odd and by property (1) of the signed index permutation  $q_{|\sigma(i)|} \geq q_{|\sigma(i+1)|}$ . Therefore  $q_{|\sigma(i)|} > q_{|\sigma(i+1)|}$  which means  $q_{|\sigma(i)|} - 1 \geq q_{|\sigma(i+1)|}$ .

- Case 3. Suppose that  $\sigma(i) > \sigma(i+1)$  and  $\varepsilon_i(\sigma) = \varepsilon_{i+1}(\sigma)$ . Then  $d_i(\sigma) = d_{i+1}(\sigma) + 1$ . In this case we need to show that  $q_{|\sigma(i)|} - 2 \geq q_{|\sigma(i+1)|}$ . Since  $\varepsilon_i(\sigma) = \varepsilon_{i+1}(\sigma)$ , then  $q_{|\sigma(i)|}$  and  $q_{|\sigma(i+1)|}$  must have the same parity and by condition (1) of the signed index permutation we have  $q_{|\sigma(i)|} \geq q_{|\sigma(i+1)|}$ . Thus we only need to prove that  $q_{|\sigma(i)|} > q_{|\sigma(i+1)|}$ . Assume by contradiction that  $q_{|\sigma(i)|} = q_{|\sigma(i+1)|}$ . Using condition (2) of the signed index permutation we conclude  $\sigma(i) < \sigma(i+1)$  which contradicts our original assumption.  $\square$

By the previous property, for every  $1 \leq i \leq n$  we can find a non-negative number  $\mu_{|\sigma(i)|}$  such that  $q_{|\sigma(i)|} = 2\mu_{|\sigma(i)|} + f_i(\sigma)$ . Define  $\gamma_{|\sigma(i)|} := f_i(\sigma)$  so that  $\gamma_i = f_{|\sigma^{-1}(i)|}(\sigma)$ .

**Proposition 3.6.** The sequences  $\gamma = (\gamma_1, \dots, \gamma_n)$  and  $\mu = (\mu_1, \dots, \mu_n)$  are sequences of non-negative integers that satisfy the following properties:

$$(1) \quad q = 2\mu + \gamma,$$

- (2)  $\mu_{|\sigma(1)|} \geq \mu_{|\sigma(2)|} \geq \cdots \geq \mu_{|\sigma(n)|}$ ,
- (3)  $\gamma_{|\sigma(1)|} \geq \gamma_{|\sigma(2)|} \geq \cdots \geq \gamma_{|\sigma(n)|}$ ,
- (4) if  $0 < i < j \leq n$  and  $\gamma_i = \gamma_j$  then  $\mathfrak{s}(q_i) \geq \mathfrak{s}(q_j)$ .

**Proof:** Properties (1)–(3) follow directly from the construction and from the properties of the numbers  $f_i(\sigma)$ . Suppose now that  $0 < i < j \leq n$  and  $\gamma_i = \gamma_j$ . This means  $f_{|\sigma^{-1}(i)|}(\sigma) = f_{|\sigma^{-1}(j)|}(\sigma)$ ; that is,  $2d_{|\sigma^{-1}(i)|}(\sigma) + \varepsilon_{|\sigma^{-1}(i)|}(\sigma) = 2d_{|\sigma^{-1}(j)|}(\sigma) + \varepsilon_{|\sigma^{-1}(j)|}(\sigma)$ . It follows that  $\varepsilon_{|\sigma^{-1}(i)|}(\sigma) = \varepsilon_{|\sigma^{-1}(j)|}(\sigma)$  and  $d_{|\sigma^{-1}(i)|}(\sigma) = d_{|\sigma^{-1}(j)|}(\sigma)$ . By definition

$$\begin{aligned} q_i &= \gamma_i + 2\mu_i = 2(\mu_i + d_{|\sigma^{-1}(i)|}(\sigma)) + \varepsilon_{|\sigma^{-1}(i)|}(\sigma), \\ q_j &= \gamma_j + 2\mu_j = 2(\mu_j + d_{|\sigma^{-1}(j)|}(\sigma)) + \varepsilon_{|\sigma^{-1}(j)|}(\sigma). \end{aligned}$$

In particular we conclude that  $q_i$  and  $q_j$  have the same parity. We need to consider two cases according to the parity of these numbers. Suppose first that  $q_i$  and  $q_j$  are even. Let  $k = |\sigma^{-1}(i)|$  and  $l = |\sigma^{-1}(j)|$ . Since  $q_i$  and  $q_j$  are even, then  $\varepsilon_k(\sigma) = \varepsilon_l(\sigma) = 0$  and this implies that  $\sigma^{-1}(i), \sigma^{-1}(j) > 0$ ; that is,  $k = \sigma^{-1}(i)$  and  $l = \sigma^{-1}(j)$ . Let's show that  $k < l$ . Assume by contradiction that  $l < k$ . Since  $d_l(\sigma) = d_k(\sigma)$  and  $l < k$ , then by property (2) of the numbers  $d_i(\sigma)$  it follows

$$j = \sigma(l) < \sigma(l+1) < \cdots < \sigma(k) = i$$

which contradicts the assumption  $i < j$ . Therefore  $0 < \sigma^{-1}(i) < \sigma^{-1}(j)$  and by property (1) of the signed index permutation we conclude  $q_i = q_{|\sigma(\sigma^{-1}(i))|} \geq q_{|\sigma(\sigma^{-1}(j))|} = q_j$ . The case where  $q_i$  and  $q_j$  are odd is handled in a similar way.  $\square$

Next we obtain a similar decomposition for the sequence  $p = (p_1, \dots, p_n)$ . To start assume  $m(\mathbf{x}, \mathbf{y}) = x^p y^q \in \mathcal{O}_n$ . This implies that  $p_i + q_i$  is even for all  $1 \leq i \leq n$ . On the other hand,  $\varepsilon_i(\sigma^{-1}) = 0$  if and only if  $k := \sigma^{-1}(i) > 0$  and this is the case if and only if  $q_i = q_{\sigma(k)}$  is even. We conclude that if  $m(\mathbf{x}, \mathbf{y}) \in \mathcal{O}_n$  then for every  $1 \leq i \leq n$

$$p_i \equiv q_i \equiv \varepsilon_i(\sigma^{-1}) \pmod{2}.$$

**Property 3.7.** Suppose that  $m(\mathbf{x}, \mathbf{y}) = x^p y^q \in \mathcal{O}_n$ . Then  $\{p_i - f_i(\sigma^{-1})\}_{i=1}^n$  is a decreasing sequence of non-negative even integers.

**Proof:** By the above comment

$$\begin{aligned} p_i - f_i(\sigma^{-1}) &= p_i - 2d_i(\sigma^{-1}) - \varepsilon_i(\sigma^{-1}) \\ &\equiv q_i - \varepsilon_i(\sigma^{-1}) \pmod{2} \\ &\equiv 0 \pmod{2}. \end{aligned}$$

This proves that  $p_i - f_i(\sigma^{-1})$  is even for all  $1 \leq i \leq n$ . On the other hand, note that  $p_n - f_n(\sigma^{-1}) = p_n - \varepsilon_n(\sigma^{-1}) \geq 0$  because  $p_n$  is odd if and only if  $\varepsilon_n(\sigma^{-1}) = 1$ . We are left to prove that  $p_i - f_i(\sigma^{-1}) \geq p_{i+1} - f_{i+1}(\sigma^{-1})$  for all  $1 \leq i \leq n-1$ . For this we consider the following cases.

- **Case 1.** Suppose that  $\sigma^{-1}(i) < \sigma^{-1}(i+1)$ . This implies  $d_i(\sigma^{-1}) = d_{i+1}(\sigma^{-1})$ . Thus in this case we need to prove that  $p_i - \varepsilon_i(\sigma^{-1}) \geq p_{i+1} - \varepsilon_{i+1}(\sigma^{-1})$ . Since  $p_i \geq p_{i+1}$  the only case we need to inspect is the case  $\varepsilon_i(\sigma^{-1}) = 1$  and  $\varepsilon_{i+1}(\sigma^{-1}) = 0$ . However, under this assumption  $p_i$  is odd and  $p_{i+1}$  is even and thus  $p_i - 1 \geq p_{i+1}$  as  $p_i \geq p_{i+1}$ .

• Case 2. Suppose that  $\sigma^{-1}(i) > \sigma^{-1}(i+1)$  and  $\varepsilon_i(\sigma^{-1}) \neq \varepsilon_{i+1}(\sigma^{-1})$ . This is only possible if  $\varepsilon_i(\sigma^{-1}) = 0$  and  $\varepsilon_{i+1}(\sigma^{-1}) = 1$ . Then  $i \in \text{Des}(\sigma^{-1})$  and  $d_i(\sigma^{-1}) = d_{i+1}(\sigma^{-1}) + 1$ . In this case we need to show that  $p_i - 1 \geq p_{i+1}$ . Note that  $p_i$  must be even and  $p_{i+1}$  must be odd and by assumption  $p_i \geq p_{i+1}$ . Therefore  $p_i > p_{i+1}$  and thus  $p_i - 1 \geq p_{i+1}$  as desired.

• Case 3. Suppose that  $\sigma^{-1}(i) > \sigma^{-1}(i+1)$  and  $\varepsilon_i(\sigma^{-1}) = \varepsilon_{i+1}(\sigma^{-1})$ . Then  $i \in \text{Des}(\sigma^{-1})$  and therefore  $d_i(\sigma^{-1}) = d_{i+1}(\sigma^{-1}) + 1$ . In this case we need to show that  $p_i - 2 \geq p_{i+1}$ . Note that  $\varepsilon_i(\sigma^{-1}) = \varepsilon_{i+1}(\sigma^{-1})$  implies that  $p_i$  and  $p_{i+1}$  have the same parity. We know that  $p_i \geq p_{i+1}$ . Therefore we only need to prove that  $p_i > p_{i+1}$ . Assume by contradiction that  $p_i = p_{i+1}$ . By assumption  $m(\mathbf{x}, \mathbf{y}) = x^p y^q \in \mathcal{O}_n$  and  $p_i = p_{i+1}$ . This implies  $\mathfrak{s}(q_i) \geq \mathfrak{s}(q_{i+1})$ . Let  $k = |\sigma^{-1}(i)|$  and  $l = |\sigma^{-1}(i+1)|$ . If  $\varepsilon_i(\sigma^{-1}) = \varepsilon_{i+1}(\sigma^{-1}) = 0$  then we obtain  $0 < l < k$  and therefore we conclude  $q_{i+1} = q_{|\sigma(l)|} \geq q_{|\sigma(k)|} = q_i$  by property (1) of the signed index permutation  $\sigma$ . Note that  $q_i$  and  $q_{i+1}$  must be even since they have the same parity as  $\varepsilon_i(\sigma^{-1}) = \varepsilon_{i+1}(\sigma^{-1}) = 0$ . Thus  $q_i = \mathfrak{s}(q_i) \geq \mathfrak{s}(q_{i+1}) = q_{i+1}$  and in turn  $q_{i+1} = q_{|\sigma(l)|} = q_{|\sigma(k)|} = q_i$ . This together with property (2) of the signed index permutation  $\sigma$  imply  $i+1 = \sigma(l) < \sigma(k) = i$  which is a contradiction. Suppose now that  $\varepsilon_i(\sigma^{-1}) = \varepsilon_{i+1}(\sigma^{-1}) = 1$ . Then  $k = -\sigma^{-1}(i)$  and  $l = -\sigma^{-1}(i+1)$  and by assumption  $\sigma^{-1}(i) > \sigma^{-1}(i+1)$ . Thus  $0 > -k > -l$ ; that is,  $0 < k < l$ . Using property (1) of the signed index permutation we conclude  $q_i = q_{|\sigma(k)|} \geq q_{|\sigma(l)|} = q_{i+1}$ . Under the given assumptions  $q_i$  and  $q_{i+1}$  must be odd and  $\mathfrak{s}(q_i) \geq \mathfrak{s}(q_{i+1})$ ; that is,  $q_i \leq q_{i+1}$ . Again we conclude that  $q_i = q_{|\sigma(k)|} = q_{|\sigma(l)|} = q_{i+1}$ . Since  $0 < k < l$  using property (2) as before we conclude that  $-i = \sigma(k) < \sigma(l) = -i-1$  deriving a contradiction in either case.  $\square$

By the previous property, for every  $1 \leq i \leq n$  we can find a non-negative integer  $\nu_i$  such that  $2\nu_i = p_i - f_i(\sigma^{-1})$ ; that is,  $p_i = 2\nu_i + f_i(\sigma^{-1})$ . Define  $\delta_i := f_i(\sigma^{-1})$ .

**Proposition 3.8.** *The sequences  $\delta = (\delta_1, \dots, \delta_n)$  and  $\nu = (\nu_1, \dots, \nu_n)$  are sequences of non-negative integers that satisfy the following properties:*

- (1)  $p = 2\nu + \delta$ ,
- (2)  $\nu_1 \geq \nu_2 \geq \dots \geq \nu_n$ ,
- (3)  $\delta_1 \geq \delta_2 \geq \dots \geq \delta_n$ ,
- (4) if  $0 < i < j \leq n$  and  $\delta_i = \delta_j$  then  $\mathfrak{s}(q_i) \geq \mathfrak{s}(q_j)$ .

**Proof:** Properties (1)–(3) follow directly from the construction, Property 3.7 and the properties of the numbers  $f_i(\sigma^{-1})$ . Suppose now that  $0 < i < j \leq n$  and  $\delta_i = \delta_j$ . This means  $f_i(\sigma^{-1}) = f_j(\sigma^{-1})$  and in turn  $d_i(\sigma^{-1}) = d_j(\sigma^{-1})$  and  $\varepsilon_i(\sigma^{-1}) = \varepsilon_j(\sigma^{-1})$ . Since  $i < j$  by property (2) of the numbers  $d_i(\sigma^{-1})$  it follows that  $\sigma^{-1}(i) < \sigma^{-1}(j)$ . Since  $\varepsilon_i(\sigma^{-1}) = \varepsilon_j(\sigma^{-1})$  then  $q_i$  and  $q_j$  have the same parity. Assume  $q_i$  and  $q_j$  are both even, then  $0 < \sigma^{-1}(i) < \sigma^{-1}(j)$  and by property (1) of the signed index permutation we have

$$q_i = q_{|\sigma(\sigma^{-1}(i))|} \geq q_{|\sigma(\sigma^{-1}(j))|} = q_j.$$

Similarly, if  $q_i$  and  $q_j$  are odd, then  $\sigma^{-1}(i) < \sigma^{-1}(j) < 0$  and thus  $-\sigma^{-1}(i) > -\sigma^{-1}(j) > 0$ . Again by property (1) of the number signed index permutation we have

$$q_i = q_{|\sigma(\sigma^{-1}(i))|} \leq q_{|\sigma(\sigma^{-1}(j))|} = q_j.$$

In either case we obtain  $\mathfrak{s}(q_i) \geq \mathfrak{s}(q_j)$ .  $\square$

**3.8. Monomial decomposition.** Suppose that  $m(\mathbf{x}, \mathbf{y}) = x^p y^q \in \mathcal{O}_n$ . We can decompose  $p = 2\nu + \delta$  and  $q = 2\mu + \gamma$  using Propositions 3.8 and 3.6. Let  $\sigma \in B_n$  be the index permutation associated to the monomial  $m(\mathbf{x}, \mathbf{y})$ . By definition  $\delta_i = f_i(\sigma^{-1})$  and  $\gamma_i = f_{|\sigma^{-1}(i)|}(\sigma)$ . Therefore

$$c_\sigma = \prod_{i=1}^n x_i^{f_i(\sigma^{-1})} y_i^{f_{|\sigma^{-1}(i)|}(\sigma)} = x^\delta y^\gamma.$$

This means that we have a decomposition

$$m(\mathbf{x}, \mathbf{y}) = x^p y^q = x^{2\nu+\delta} y^{2\mu+\gamma} = x^{2\nu} y^{2\mu} c_\sigma.$$

Given a sequence of integers  $i = (i_1, \dots, i_n)$  let  $\Sigma_n(i)$  denote the stabilizer of  $i$  under the permutation action of  $\Sigma_n$ ; that is,  $\Sigma_n(i)$  is the subgroup of elements in  $\Sigma_n$  that fix  $i$ . Define

$$m_{2\nu}(\mathbf{x}) = \sum_{[\alpha] \in \Sigma_n / \Sigma_n(\nu)} x^{2\alpha(\nu)}, \text{ and } m_{2\mu}(\mathbf{y}) = \sum_{[\beta] \in \Sigma_n / \Sigma_n(\mu)} y^{2\beta(\mu)}.$$

By definition  $m_{2\nu}(\mathbf{x})$  and  $m_{2\mu}(\mathbf{y})$  are symmetric polynomials on the variables  $x_1^2, \dots, x_n^2$  and  $y_1^2, \dots, y_n^2$  respectively. In particular  $m_{2\nu}(\mathbf{x}) m_{2\mu}(\mathbf{y}) \in R^{B_n}$ . In the same way as in [5, Proposition 3.2] the polynomial  $m_{2\nu}(\mathbf{x}) m_{2\mu}(\mathbf{y}) \rho(c_\sigma)$  can be decomposed as we prove next.

**Theorem 3.9.** *Suppose that  $m(\mathbf{x}, \mathbf{y}) \in \mathcal{O}_n$  and let  $\sigma \in B_n$  be the corresponding singed index permutation. Then*

$$m_{2\nu}(\mathbf{x}) m_{2\mu}(\mathbf{y}) \rho(c_\sigma) = k_{m(\mathbf{x}, \mathbf{y})} \rho(m(\mathbf{x}, \mathbf{y})) + \sum_{n(\mathbf{x}, \mathbf{y}) \not\preceq m(\mathbf{x}, \mathbf{y})} k_{n(\mathbf{x}, \mathbf{y})} \rho(n(\mathbf{x}, \mathbf{y})).$$

Here  $n(\mathbf{x}, \mathbf{y}) \in \mathcal{O}_n$  runs through the collection of ordered monomials with same total degree as  $m(\mathbf{x}, \mathbf{y})$  with  $n(\mathbf{x}, \mathbf{y}) \not\preceq m(\mathbf{x}, \mathbf{y})$ , and  $k_{m(\mathbf{x}, \mathbf{y})} > 0$ ,  $k_{n(\mathbf{x}, \mathbf{y})}$  are constants.

**Proof:** Using the definitions we have

$$\begin{aligned} m_{2\nu}(\mathbf{x}) m_{2\mu}(\mathbf{y}) \rho(c_\sigma) &= \rho(m_{2\nu}(\mathbf{x}) m_{2\mu}(\mathbf{y}) c_\sigma) = \sum_{[\alpha] \in \Sigma_n / \Sigma_n(\nu)} \sum_{[\beta] \in \Sigma_n / \Sigma_n(\mu)} \rho(x^{2\alpha\nu} y^{2\beta\mu} c_\sigma) \\ &= \sum_{[\alpha] \in \Sigma_n / \Sigma_n(\nu)} \sum_{[\beta] \in \Sigma_n / \Sigma_n(\mu)} \rho(x^{2\alpha\nu+\delta} y^{2\beta\mu+\gamma}). \end{aligned}$$

Fix  $\alpha, \beta \in \Sigma$  and consider  $n(\mathbf{x}, \mathbf{y}) \in \mathcal{O}_n$  the unique monomial such that  $\rho(n(\mathbf{x}, \mathbf{y})) = \rho(x^{2\alpha\nu+\delta} y^{2\beta\mu+\gamma})$ . To prove the theorem we need to prove that  $n(\mathbf{x}, \mathbf{y}) \preceq m(\mathbf{x}, \mathbf{y})$ . Let  $[\Sigma_n(\delta)]$  denote the image of  $\Sigma_n(\delta)$  in  $\Sigma_n / \Sigma_n(\nu)$  under the natural map. By parts (2) and (3) of Proposition 3.8 it follows that if  $[\alpha] \in [\Sigma_n(\delta)]$  then  $\mathfrak{o}(2\alpha\nu + \delta) = \mathfrak{o}(p)$ , and if  $[\alpha] \notin [\Sigma_n(\delta)]$  then  $\mathfrak{o}(2\alpha\nu + \delta) <_\ell \mathfrak{o}(p)$ . Similarly, if  $[\Sigma_n(\gamma)]$  denote the image of  $\Sigma_n(\gamma)$  in  $\Sigma_n / \Sigma_n(\mu)$  then by parts (2) and (3) of Proposition 3.6 it follows that if  $[\beta] \in [\Sigma_n(\gamma)]$  then  $\mathfrak{o}(2\beta\mu + \gamma) = \mathfrak{o}(q)$ , and if  $[\beta] \notin [\Sigma_n(\gamma)]$  then  $\mathfrak{o}(2\beta\mu + \gamma) <_\ell \mathfrak{o}(q)$ . With this in mind we have the following cases.

- Case 1. Suppose that  $[\alpha] \notin [\Sigma_n(\delta)]$  or  $[\beta] \notin [\Sigma_n(\gamma)]$ . If  $[\alpha] \notin [\Sigma_n(\delta)]$  then by the previous comment  $\mathfrak{o}(2\alpha\nu + \delta) <_\ell \mathfrak{o}(p)$  and if  $[\alpha] \in [\Sigma_n(\delta)]$  but  $\beta \notin [\Sigma_n(\gamma)]$  then

$\mathfrak{o}(2\alpha\nu + \delta) = \mathfrak{o}(p)$  but  $\mathfrak{o}(2\beta\mu + \gamma) <_\ell \mathfrak{o}(q)$ . In either case we conclude

$$\mathfrak{o}(n(\mathbf{x}, \mathbf{y})) = (\mathfrak{o}(2\alpha\nu + \delta), \mathfrak{o}(2\beta\mu + \gamma)) <_\ell (\mathfrak{o}(p), \mathfrak{o}(q)) = \mathfrak{o}(m(\mathbf{x}, \mathbf{y})).$$

It follows that in this case  $n(\mathbf{x}, \mathbf{y}) \not\preceq m(\mathbf{x}, \mathbf{y})$ .

• Case 2. Suppose that  $[\alpha] \in [\Sigma_n(\delta)]$  and  $[\beta] \in [\Sigma_n(\gamma)]$ . We can assume without loss of generality that  $\alpha \in \Sigma_n(\delta)$  and  $\beta \in \Sigma_n(\gamma)$ . Then

$$x^{2\alpha\nu + \delta} y^{2\beta\mu + \gamma} = x^{\alpha(2\nu + \delta)} y^{\beta(2\mu + \gamma)} = x^{\alpha(p)} y^{\beta(q)}.$$

Therefore  $n(\mathbf{x}, \mathbf{y}) = x^p y^{\pi\alpha^{-1}\beta(q)}$  for some  $\pi \in \Sigma_n$ . Note that the permutation  $\pi$  has to stabilize  $p$  and  $\delta$ , thus in particular  $\pi \in \Sigma_n(\delta)$ . In this case

$$\mathfrak{o}(n(\mathbf{x}, \mathbf{y})) = \mathfrak{o}(x^{\alpha(p)} y^{\beta(q)}) = \mathfrak{o}(m(\mathbf{x}, \mathbf{y})).$$

To prove that  $n(\mathbf{x}, \mathbf{y}) \preceq m(\mathbf{x}, \mathbf{y})$  we need to show that  $(p, \mathfrak{s}(\pi\alpha^{-1}\beta(q))) \leq_\ell (p, \mathfrak{s}(q))$ ; that is, we need to prove that  $\mathfrak{s}(\pi\alpha^{-1}\beta(q)) \leq_\ell \mathfrak{s}(q)$ . If  $\mathfrak{s}(\pi\alpha^{-1}\beta(q)) = \mathfrak{s}(q)$  there is nothing to prove. Assume that  $\mathfrak{s}(\pi\alpha^{-1}\beta(q)) \neq \mathfrak{s}(q)$  and let  $1 \leq k \leq n$  be the smallest integer such that  $\mathfrak{s}(q_{\pi\alpha^{-1}\beta(k)}) \neq \mathfrak{s}(q_k)$ . We need to show that  $\mathfrak{s}(q_{\pi\alpha^{-1}\beta(k)}) < \mathfrak{s}(q_k)$ . Since  $\beta \in \Sigma_n(\gamma)$ , then by Proposition 3.6 part (4) we have that if  $i < \beta(i)$  then  $\mathfrak{s}(q_i) \geq \mathfrak{s}(q_{\beta(i)})$ . Similarly, since  $\alpha, \pi \in \Sigma_n(\delta)$  by Proposition 3.8 part (4) whenever  $i < \pi\alpha^{-1}(i)$  then  $\mathfrak{s}(q_i) \geq \mathfrak{s}(q_{\pi\alpha^{-1}(i)})$ . Using this we can see that  $\mathfrak{s}(q_k) > \mathfrak{s}(q_{\pi\alpha^{-1}\beta(k)})$ .

In either case we conclude that  $n(\mathbf{x}, \mathbf{y}) \preceq m(\mathbf{x}, \mathbf{y})$ .  $\square$

**3.9. Main theorem.** Finally we are ready to prove the main theorem of this article.

**Theorem 3.10.** *Suppose that  $n \geq 1$ . Then the collection  $\{\rho(c_\sigma)\}_{\sigma \in B_n}$  forms a free basis of  $M^{B_n}$  as a module over  $R^{B_n}$ .*

**Proof:** Let's show first that  $\{\rho(c_\sigma)\}_{\sigma \in B_n}$  generates  $M^{B_n}$  as a module over  $R^{B_n}$ . It suffices to show that for every  $m(\mathbf{x}, \mathbf{y}) \in \mathcal{O}_n$  the polynomial  $\rho(m(\mathbf{x}, \mathbf{y}))$  is generated by the different  $\rho(c_\sigma)$ . Fix  $m(\mathbf{x}, \mathbf{y}) \in \mathcal{O}_n$  and let  $\sigma$  be the corresponding signed index permutation. By the previous theorem we have

$$m(\mathbf{x}, \mathbf{y}) = l_{m(\mathbf{x}, \mathbf{y})} m_{2\nu}(\mathbf{x}) m_{2\mu}(\mathbf{y}) \rho(c_\sigma) + \sum_{n(\mathbf{x}, \mathbf{y}) \not\preceq m(\mathbf{x}, \mathbf{y})} l_{n(\mathbf{x}, \mathbf{y})} n(\mathbf{x}, \mathbf{y}),$$

for some constants  $l_{n(\mathbf{x}, \mathbf{y})}$  and monomials  $n(\mathbf{x}, \mathbf{y}) \not\preceq m(\mathbf{x}, \mathbf{y})$  of same total degree. Iterating this process on the monomials  $n(\mathbf{x}, \mathbf{y}) \not\preceq m(\mathbf{x}, \mathbf{y})$  as many times as necessary we see that we can write  $m(\mathbf{x}, \mathbf{y})$  as a linear combination of the elements  $\{\rho(c_\sigma)\}_{\sigma \in B_n}$  with coefficients in  $R^{B_n}$ . (Note that this process must terminate after finitely many stages as there are only finitely many monomials  $n(\mathbf{x}, \mathbf{y})$  of same total degree as  $m(\mathbf{x}, \mathbf{y})$ ). This proves that  $\{\rho(c_\sigma)\}_{\sigma \in B_n}$  generates  $M^{B_n}$  as a  $R^{B_n}$ -module. On the other hand, note that  $M^{B_n}$  is a bigraded ring over  $\mathbb{Q}$  with

$$\text{bideg}(x^p y^q) = (|p|, |q|),$$

where  $|p| = p_1 + \dots + p_n$ . With this graduation for every  $\sigma \in B_n$  we have that the polynomial  $\rho(c_\sigma)$  is homogeneous and

$$\text{bideg}(\rho(c_\sigma)) = (\text{fmaj}(\sigma^{-1}), \text{fmaj}(\sigma)).$$

Let  $P_{M^{B_n}}(s, t)$  denote the bigraded Hilbert series of the bigraded ring  $M^{B_n}$ . Using [4, Theorem 3] we conclude that the series  $P_{M^{B_n}}(s, t)$  is given by

$$P_{M^{B_n}}(s, t) = \frac{\left( \sum_{\sigma \in B_n} s^{\text{fmaj}(\sigma^{-1})} t^{\text{fmaj}(\sigma)} \right)}{\prod_{i=1}^n (1 - s^{2i})(1 - t^{2i})}.$$

This together with [5, Theorem 1.4] show that  $\{\rho(c_\sigma)\}_{\sigma \in B_n}$  is a free basis of  $M^{B_n}$  as module over  $R^{B_n}$ .  $\square$

As in the previous cases this theorem also has a geometric application. Let  $B(2, Sp(n))$  be the geometric realization of the simplicial space obtained by considering the space of commuting  $k$ -tuples in  $Sp(n)$ ,  $B_k(2, Sp(n)) = \text{Hom}(\mathbb{Z}^k, Sp(n))$ . In [1] it is proved that the signed diagonal descent monomials can be used to obtain an explicit basis of  $H^*(B(2, Sp(n)); \mathbb{Q})$  seen as a module over  $H^*(BSp(n); \mathbb{Q})$ .

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